Transmission Line analysis following an analogy to a linear array of masses connected by springs.

Sept. 11, 2025

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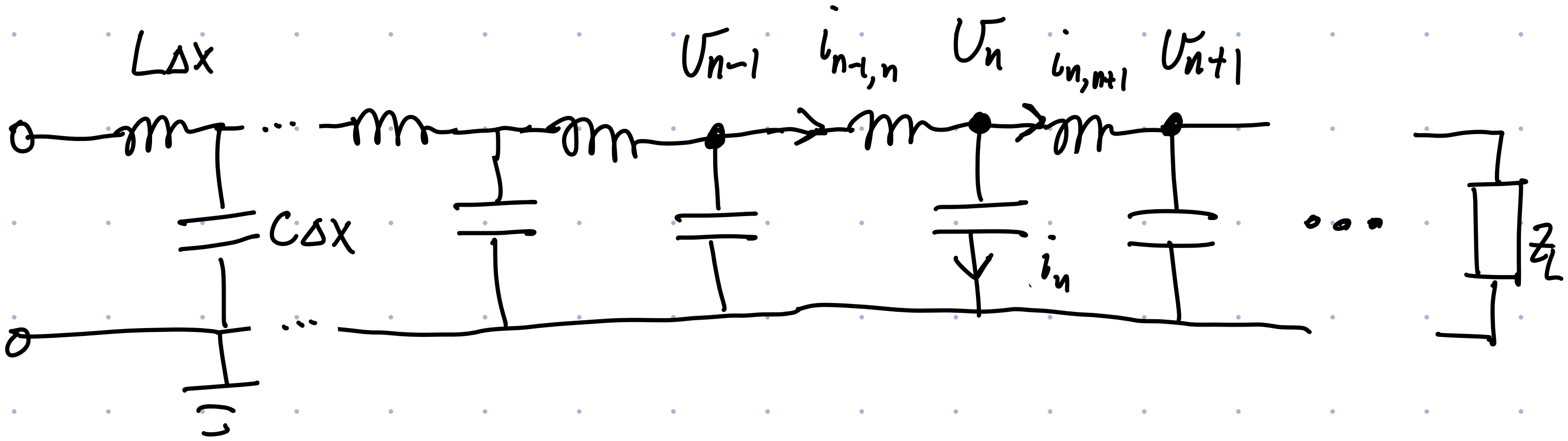


Junction rule at V_n:

$$i_{n-1,n} = i_n + i_{n,n+1}$$

$$\therefore\ i_n = i_{n-1,n} - i_{n,n+1}$$

$$\therefore\ \dot{q}_n = i_{n-1,n} - i_{n,n+1}$$

$$\therefore\ \ddot{q}_n = \frac{d\,i_{n-1,n}}{dt} - \frac{d\,i_{n,n+1}}{dt} \quad (1)$$

Charge on cap.

$$q_n = C\Delta x\, V_n$$

$$\therefore\ i_n = \dot{q}_n$$

Next, if we go from V_{n-1} to V_n through the inductance, we require:

For current i_-: $\Delta V_{L-} = V_n - V_{n-1} = -L\Delta x \dfrac{d i_{n-1,n}}{dt}$

$$\therefore \frac{d i_{n-1,n}}{dt} = -\left(\frac{V_n - V_{n-1}}{L\Delta x}\right) = \frac{q_{n-1} - q_n}{LC\Delta x^2} \quad (2)$$

Likewise, for current i_+:

$$\Delta V_{L+} = V_{n+1} - V_n = -L\Delta x \frac{d i_{n,n+1}}{dt}$$

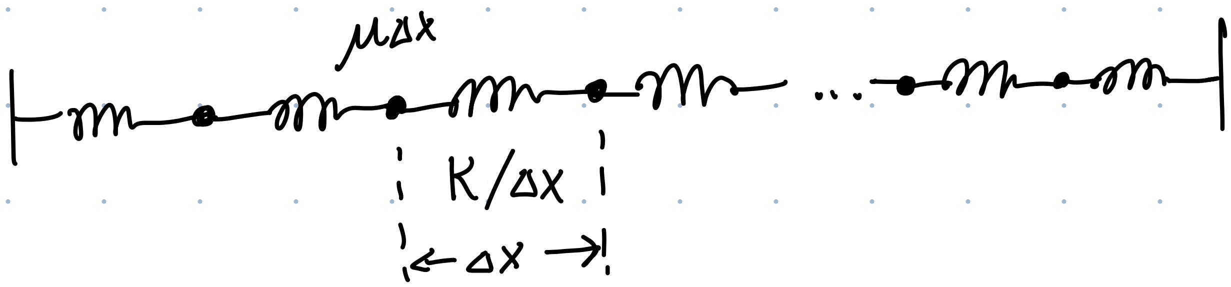
$$\therefore \frac{d i_{n,n+1}}{dt} = \frac{q_n - q_{n+1}}{LC\Delta x^2} \quad (3)$$

Sub (2) & (3) into (1)

$$\ddot{q}_n = \frac{q_{n-1} - q_n}{LC\Delta x^2} - \frac{q_n - q_{n+1}}{LC\Delta x^2}$$

$$\boxed{\therefore \quad LC\Delta x^2 \ddot{q}_n = q_{n-1} - 2q_n + q_{n+1}} \quad ⓐ$$

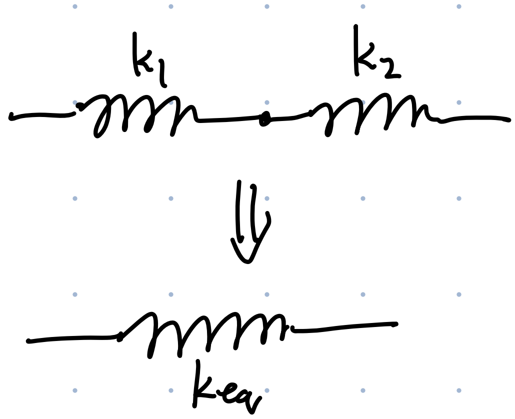
Before continuing, let's make an analogy to a mechanical system → a linear array of pt. masses connected by springs.



Here, μ per unit length mass $(m = \mu\Delta x)$.

The stiffness, on the other hand, must scale as $\frac{1}{\Delta x}$.

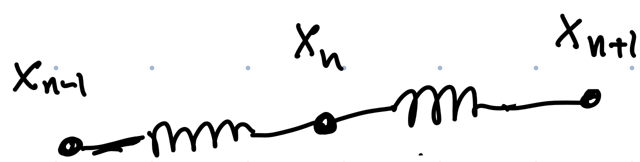
Why? Springs in series combine s.t. their stiffnesses add in inverse.



$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$\therefore$ As we decrease Δx, we are combining more and more springs in series. To keep the overall stiffness of the system constant, we must compensate by sealing $k = \frac{K}{\Delta x}$

FBD of a single pt. mass.



$$\mu \Delta x \ddot{x}_n = -\frac{K}{\Delta x}(x_n - x_{n-1}) + \frac{K}{\Delta x}(x_{n+1} - x_n)$$
$$= \frac{K}{\Delta x}(x_{n-1} - 2x_n + x_{n+1})$$

$$\boxed{\therefore \frac{\mu}{K}\Delta x^2 \ddot{x}_n = x_{n-1} - 2x_n + x_{n+1}}$$ (b)

Notice that a and b have identical forms!

The TL model is analogous to the mass-spring system which itself can be used to model the physics of a 1-D lattice in which there are couplings between adjacent lattice points.

To complete the analogy, note that

$$V_L = L\frac{di}{dt} = L\frac{d^2q}{dt^2} \quad \text{and} \quad F = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

$\therefore \; L \longleftrightarrow m$

L is inertia for i, like m is inertia for v.

$$V_c = \frac{q}{C} \quad \text{and} \quad F_s = kx$$

$$\frac{1}{C} \longleftrightarrow k$$

$\frac{1}{C}$ couples adjecent inductors

Mass-spring system: When all springs in equil, all masses at rest (no K.E.). When a mass is displaced (x_n out of equil), the springs apply a net force to restore equil. ($\dot{x}_n \neq 0$)

LC Transmission line: When all caps at equil. charge, voltage diff. across L is zero & $\dot{q}_n = 0$. When a cap is out of equil, it absorbs or releases charge s.t. $\dot{q}_n \neq 0$ and we get changing currents through the inductors.

Back to (a):

$$\boxed{\therefore \; LC\Delta x^2 \ddot{q}_n = q_{n-1} - 2q_n + q_{n+1}} \quad (a)$$

Consider the Taylor expansion of $f(x)$ about $x=a$:

$$f(x) \approx f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \ldots$$

Now, consider $f(x+h)$ and expand about x

Define $u = x+h$. Thus, $x = u-h$

$$f(u) \approx f(\underbrace{u-h}_{x}) + (\cancel{u}-(\cancel{u}-h))f'(\underbrace{u-h}_{x}) + \frac{1}{2}(\cancel{u}-(\cancel{u}-h))^2 f''(\underbrace{u-h}_{x}) + \dots$$

$$\boxed{\therefore f(x+h) \approx f(x) + h f'(x) + \frac{1}{2}h^2 f''(x) + \dots}$$

Likewise,

$$\boxed{f(x-h) \approx f(x) - h f'(x) + \frac{1}{2}h^2 f''(x) + \dots}$$

$$\therefore f(x-h) + f(x+h) \approx 2f(x) + h^2 f''(x).$$

$$\therefore f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

In ⓐ, we identify $q_n = q(x)$

$$\left.\begin{aligned} q_{n-1} &= q(x-\Delta x) \\ q_{n+1} &= q(x+\Delta x) \end{aligned}\right\}$$ Δx is the segment length/lattice constant.

$$\therefore \frac{q_{n-1} - 2q_n + q_{n+1}}{\Delta x^2} \approx \frac{\partial^2 q(x)}{\partial x^2}$$

The equality becomes exact in the continuum limit $\Delta x \to 0$, and $q_n \to q(x)$.

$\therefore$ ⓐ becomes $$LC\,\ddot{q}(x) \approx \frac{\partial^2 q(x)}{\partial x^2}$$

$q(x)$ is the capacitor voltage. $\therefore$ The voltage across the parallel TL conductors is $v = \frac{q}{C}$

$$\therefore \frac{\ddot{q}(x)}{C} \to \ddot{v}(x) \qquad \frac{\partial^2 q(x)/C}{\partial x^2} \to \frac{\partial^2 v(x)}{\partial x^2}$$

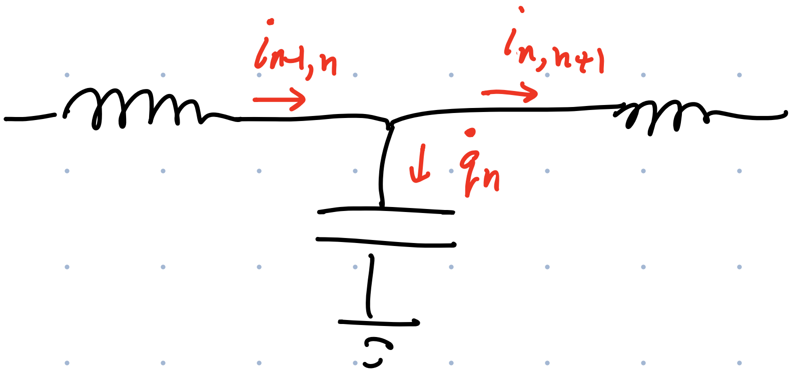
We have recovered the first wave equation

$$\boxed{\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}} \quad (4)$$

To get the 2nd wave equation, return to (3):

$$\frac{d\, i_{n,n+1}}{dt} = \frac{q_n - q_{n+1}}{LC\Delta x^2}$$

$$\therefore \quad LC \frac{d^2 i_{n,n+1}}{dt^2} = \frac{\dot{q}_n - \dot{q}_{n+1}}{\Delta x^2}$$



$$i_{n-1,n} = \dot{q}_n + i_{n,n+1}$$

$$\therefore \quad \dot{q}_n = i_{n-1,n} - i_{n,n+1}$$

$$\dot{q}_{n+1} = i_{n,n+1} - i_{n+1,n+2}$$

$$\therefore \quad \frac{\dot{q}_n - \dot{q}_{n+1}}{\Delta x^2} = \frac{i_{n-1,n} - i_{n,n+1} - (i_{n,n+1} - i_{n+1,n+2})}{\Delta x^2}$$

$$= \frac{i_{n-1,n} - 2 i_{n,n+1} + i_{n+1,n+2}}{\Delta x^2}$$

$$= \frac{\partial^2 i(x)}{\partial x^2} \text{ in the } \Delta x \to 0 \text{ limit.}$$

$$\therefore \quad \boxed{\frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2}}$$

5 We've now recovered the wave equations we found in class. The rest of the analysis proceeds as presented in the class notes.

Let's return to the analogous mass-spring system and show how the discrete TL-like analysis leads to phonon dispersion relations found in solid-state physics (PHYS 474).

$$\boxed{\therefore \frac{\mu}{K}\Delta x^2 \ddot{x}_n = x_{n-1} - 2x_n + x_{n+1}} \quad (b)$$

$\mu \Delta x \to m$

$\frac{\Delta x}{K} \to \underline{K}$

$$\therefore \quad m\ddot{x}_n = \underline{K}(x_{n-1} - 2x_n + x_{n+1})$$

Assume $x_n \approx A e^{j(\omega t - kna)}$

where I've take $\Delta x = a$ to be the lattice const of the linear chain.

Note:

$\underline{K} \to$ spring const.

$k \to \frac{2\pi}{\lambda}$

$$\ddot{x}_n = -\omega^2 x_n$$

$$\therefore\ -m\omega^2 \cancel{A}\cancel{e^{j\omega t}}\cancel{e^{-jkna}} = \underline{K}\Big[\cancel{A}\cancel{e^{j\omega t}}\cancel{e^{-jkna}}\, e^{jka} - 2\cancel{A}\cancel{e^{j\omega t}}\cancel{e^{-jkna}} + \cancel{A}\cancel{e^{j\omega t}}\cancel{e^{-jkna}}\, e^{-jka}\Big]$$

$$\therefore\ -m\omega^2 = \underline{K}\Big[\underbrace{e^{jka} + e^{-jka}}_{2\cos ka} - 2\Big]$$

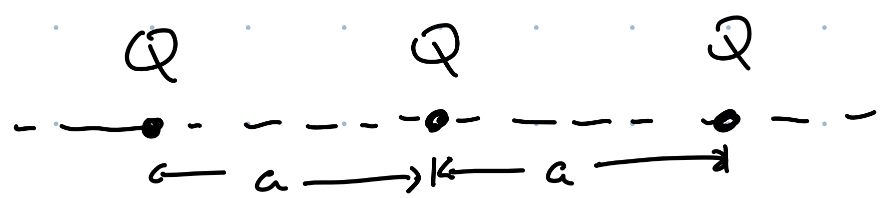
$$-m\omega^2 = -2\underline{K}\big[1-\cos ka\big]$$

usually, use the identity $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$

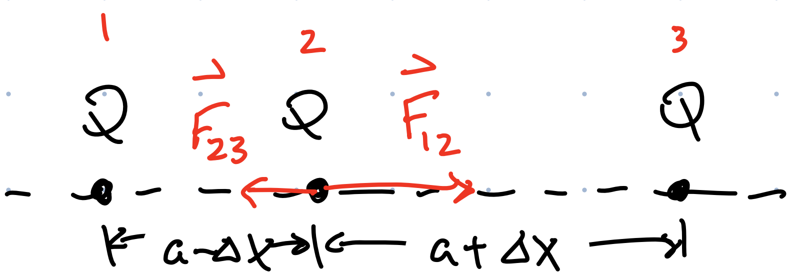
$$\boxed{\therefore\ \omega^2 = 4\frac{\underline{K}}{m}\sin^2\left(\frac{ka}{2}\right)} \quad (6)$$

This is the phonon (lattice vibration) dispersion relation.

Why is $\underline{K}$ the right model? Consider an array of 3 ions of charge Q in a lattice.



In equil., the force on the middle charge due to its neighbours is zero. Now, displace centre charge by $\Delta x \ll a$.



The net force on 2 is

Force in opp. dir'n of displacement Δx. i.e. restoring force

$$-\left[\frac{k_e Q^2}{(a-\Delta x)^2} - \frac{k_e Q^2}{(a+\Delta x)^2}\right]$$

$$= -\frac{k_e Q^2}{a^2}\left[\underbrace{\left(1-\frac{\Delta x}{a}\right)^{-2}}_{\approx 1+2\frac{\Delta x}{a}} - \underbrace{\left(1+\frac{\Delta x}{a}\right)^{-2}}_{\approx 1-\frac{2\Delta x}{a}}\right]$$

$$\approx -\frac{k_e Q^2}{a^2}\left(\frac{4\Delta x}{a}\right) = -\left(\frac{4 k_e Q^2}{a^3}\right)\Delta x$$

Of the form $F_s = -\underline{K}\,\Delta x$

w/ $\underline{K} = \dfrac{4 k_e Q^2}{a^3}$

Return to 6.

$$\boxed{\omega^2 = 4\frac{K}{m}\sin^2\left(\frac{ka}{2}\right)}$$

Consider the case in which $\lambda \gg a$. In this case, the long wavelength phonons don't resolve the discrete structure of the lattice. It, instead, sees a continuous string.

when $\lambda \gg a$, $ka = \frac{2\pi a}{\lambda} \ll 1$

and $\sin^2\left(\frac{ka}{2}\right) \approx \left(\frac{ka}{2}\right)^2$

$$\therefore \omega^2 = \frac{4K}{m}\frac{(ka)^2}{4} = \frac{K}{m}\left(\frac{2\pi}{\lambda}\right)^2 a^2$$

$$= \frac{(Ka)}{(m/a)}\left(\frac{2\pi}{\lambda}\right)^2$$

$\frac{m}{a} = \mu$ the linear mass density

$Ka = T$ the tension in the string.

$$(\cancel{2\pi} f)^2 = \frac{T}{\mu}\left(\frac{\cancel{2\pi}}{\lambda}\right)^2$$

$$\boxed{\lambda f = c = \sqrt{\frac{T}{\mu}}}$$

, where c is the speed of wave on the stretched string. Increasing the tension on the string increases the wave speed.